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EXTENSIONS OF A DYNAMIC STOCK PORTFOLIO MODEL
WITH RESPECT TO SWEDISH TAX LEGISLATION

by

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*Grubbström, R. W. and J. Lundquist, A Dynamic Stock Portfolio with Respect to Recent Swedish Tax Legislation, Proceedings, 3rd Symposium on Operations Research, Mannheim 1978.

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0. ABSTRACT

This report extends a recent model proposed by Grubbström and Lundquist [2] regarding optimal purchasing and selling policies for common stock, when taking recent Swedish tax legislation into account. In that model it was assumed that transactions only took place at the end of each year, that only one kind of stock was available and that the stockholder held no initial stock at the beginning of the process. These three limitations are relaxed in the present report. Although the model is based on Swedish tax legislation, similar rules apply in other countries and the model might therefore provide a basic frame for developing models adjusted to the specific legislation of other nations.

1. INTRODUCTION

The basic rules governing the Swedish taxation of capital gains on common stock sold are as follows. Shares are divided into two categories. Those in the possession of the stockholder for two years or more are called <u>older stock</u> and those in possession for a shorter period <u>younger stock</u>. Taxable profits on older stock are reduced to 40 per cent of the difference between price obtained and purchase cost, whereas profits on younger stock are not subject to such a reduction. Purchase cost is evaluated according to a number of rules, the principle for younger stock being the actual price paid and for older stock the average price paid for all older stock of the same kind in the possession of the stockholder at the time of sales. In the latter case the average price is to be adjusted for previous sales from the same common portfolio. Additional supplementary rules also apply, but are disregarded here. Transaction costs at the time of purchase are included in the purchase cost and similar costs at the time of sales deducted from sales price.

The average cost rule has been so unmanageable that it is presently under revision and it also appears that tax authorities as well as the public are neither able to comply with nor to comprehend this rule or its implications. In the model to be analyzed below we assume that the actual purchase price is the purchase cost for older as well as younger shares, a principle that presumably would be accepted by tax courts and also coincides with the average cost rule in cases when the stockholder always sells his entire possession of stock of the same kind.

From capital gains in a given year different deduction opportunities may be available. If a loss is incurred on older shares sold 40 per cent of this loss is deductible from other capital gains the same year. If reduced profits on older shares exceed reduced losses on older shares, a standard deduction up to a maximum of 1,000 Sw.Cr. is permitted on the remaining net profits on older shares the same year. Losses from sales of younger stock may be deducted from profits on older stock and vice versa. A total net loss is not deductible from other income the same year (except from other capital gains on real estate etc. not to be considered here) but may be deducted from profits in either one of the six years following, provided that at that time all current deduction opportunities have first been exhausted and that the loss could not have been used in the year it occurred.

Other assumptions to be applied in the present model are the following. The tax rate is assumed to be independent of income and constant for the entire process studied. No consideration is taken to wealth tax, to dividends or to transaction costs. At the beginning of the process the stockholder has a given portfolio, possibly diversified with respect to age and composition, and also a given amount of liquid assets. No inflow of additional funds, such as borrowing or savings from other income, is included in the model, nor the opportunity of shortselling (to which special additional tax rules apply). Liquid assets at any time may be put into short-term investments at a constant given after-tax opportunity rate of interest. No other competing investments are considered. Tax is paid at the end of each year in which the corresponding gains were obtained.

The objective of the stockholder is assumed to be to maximize his wealth at a given future point of time. Wealth is evaluated as the cash balance at that date, or equivalently, as his actual cash balance added to the market value of his portfolio deducted by all latent tax liabilities.

The stockholder is assumed to have complete knowledge of the prices of stock of all kinds and at all dates. All continuous points of time and time intervals are measured in fractions of years, reducing notational complexity.

2. NOTATIONS AND CONVENTIONS

The following basic notations will be used:

p_t^l = market price of stock of kind l at time t. $p_t^l > 0$ for all t and $l = 1,2, \dots N$.

$dF_{\tau t}^l$ = density of stock of kind l purchased at time τ and sold at time t, i.e. incremental volume purchased in interval $[\tau, \tau+d\tau]$ and sold in interval $[t, t+dt]$.

C_t = liquid assets at time t.

ρ = after-tax continuous rate of interest for short-term opportunity investments.

s = tax rate applicable to capital gains.

$$a_{\tau t} = \begin{cases} 1, & \text{for } t-\tau < 2 \\ .4, & \text{for } t-\tau \geq 2 \end{cases}$$ = tax reduction factor for stock purchased at τ and sold at time t.

$$b_{\tau t} = \begin{cases} 1, & \text{for } t-\tau < 2 \\ 0, & \text{for } t-\tau \geq 2 \end{cases}$$ = stock age selection factor.

T = total time period considered.

dG_τ^l = initial density of stock held at time $t=0$ of kind l purchased in interval $[\tau, \tau+d\tau]$, $\tau \leq 0$.

v_k = standard deduction available from profits on older shares in year k.

z_k = other deductions available in year k due to current losses.

x_{jk} = loss deducted in year k brought forward from year j in which it occurred.

Other notations will be introduced as the need arises. All variables given, with the exception of purchase time τ, are assumed to be non-negative.

The following additional conventions will be adopted:

a. $dF^{l}_{\tau t} = 0$, for $\tau \geqq t$ and for $t \leqq 0$. This condition prevents short-selling when $\tau > t$. When $\tau = t$ transactions would have no effect on wealth or on liquidity and are therefore omitted. Sales taken place at $t \leqq 0$ belong to the past and are beyond influence.

b. $x_{jk} = 0$, for $k-j > 6$ and $j \geqq k$. This deduction is limited to being positive only in periods in which it may be used.

c. $a^{+} = Max\{0, a\}$, for any variable a.

d. When integrals are used they will be interpreted in the Laplace-Stieltje sense, and derivatives may be generalized derivatives involving Dirac impulses etc.

e. Index t > Index τ. Typically t will denote a time of sale and τ a time of purchase.

f. As a convention we let continuous time index t belong to year k i.e. $t \in [k-1, k]$, except where not otherwise stated.

g. When not otherwise denoted, summations take place over all index values defined, taking (b) above into consideration.

3. OBJECTIVE FUNCTION, DEFINITIONS AND CONSTRAINTS

The function C_T is to be maximized by an optimal choice of the non-negative variables $dF^l_{\tau t}$, v_k, z_k and x_{jk} subject to a number of constraints. The following profit and loss measures are defined.

Gross reduced current profits on older shares in year k.

$$\hat{m}_k = \sum_l \int_{t=k-1}^{k} \int_{\tau=-\infty}^{t} a_{\tau t}(1 - b_{\tau t})(p^l_t - p^l_\tau)^+ \, dF^l_{\tau t} \qquad (1)$$

Gross current profits on younger shares in year k.

$$\hat{n}_k = \sum_l \int_{t=k-1}^{k} \int_{\tau=-\infty}^{t} a_{\tau t} b_{\tau t}(p^l_t - p^l_\tau)^+ \, dF^l_{\tau t} \qquad (2)$$

Gross reduced current loss on older shares in year k.

$$\bar{m}_k = \sum_l \int_{t=k-1}^{k} \int_{\tau=-\infty}^{t} a_{\tau t}(1 - b_{\tau t})(p^l_\tau - p^l_t)^+ \, dF^l_{\tau t} \qquad (3)$$

Gross current loss on younger shares in year k.

$$\bar{n}_k = \sum_l \int_{t=k-1}^{k} \int_{\tau=-\infty}^{t} a_{\tau t} b_{\tau t}(p^l_\tau - p^l_t)^+ \, dF^l_{\tau t} \qquad (4)$$

Taxable capital gains in year k.

$$\hat{M}_k = [(\hat{m}_k - \bar{m}_k - v_k)^+ - (\bar{m}_k - \hat{m}_k)^+ + \hat{n}_k - \bar{n}_k]^+ - \sum_j x_{jk} =$$

$$= \hat{m}_k + \hat{n}_k - v_k - z_k - \sum_j x_{jk} \qquad (5)$$

Net loss in year k.

$$\bar{M}_k = [(\bar{m}_k - \hat{m}_k)^+ - (\hat{m}_k - \bar{m}_k - v_k)^+ + \bar{n}_k - \hat{n}_k]^+ = \bar{m}_k + \bar{n}_k - z_k \qquad (6)$$

For year zero and before, $\bar{M}_k$ is given and interpreted as at $t=0$ remaining net losses from previous years, not yet having been used for deduction purposes. Only values of $\bar{M}_k$ for $k \geq -5$ need be taken into account. According to legislation the following constraints apply:

Standard deduction, older shares.

$$v_k \leqq (\hat{m}_k - \bar{m}_k)^+ \tag{7}$$

$$v_k \leqq 1000 \tag{8}$$

Non-negativity of profits shown.

$$\hat{M}_k \geqq 0 \tag{9}$$

Limitation on older losses brought forward from year k.

$$\bar{M}_k \geqq \sum_l x_{kl} \qquad (k \geqq -5,\ l \geqq 1) \tag{10}$$

Priority rules.

$$v_k(\hat{m}_k - \bar{m}_k - v_k) \geqq 0 \tag{11}$$

$$\sum_j x_{jk} (\bar{m}_k + \bar{n}_k - z_k) \leqq 0 \tag{12}$$

Constraint (11) allows the standard deduction only to be positive up to a maximum of $\hat{m}_k - \bar{m}_k$ provided this difference is positive and (12) allows losses $\sum_j x_{jk}$ only to be used in year k if all current losses have been deducted, i.e. only when $z_k = \bar{m}_k + \bar{n}_k$. Apart from the previous inequalities we require liquid assets to be non-negative:

$$C_t \geqq 0 \tag{13}$$

and that initial stockholdings sold will not exceed initial supply:

$$\int_{t=0}^{T} dF^{l}_{\tau t} \leqq dG^{l}_{\tau} \qquad (\tau \leqq 0) \tag{14}$$

The development of liquid assets over time during a particular year k is governed by the differential equation:

$$dC = \rho\ C\ dt + \sum_{l} p^{l}_{t} \left(\int_{\tau=-\infty}^{t} dF^{l}_{\tau t} - \int_{x=t}^{T} dF^{l}_{tx} \right) = \rho\ C\ dt + dh_{t} \tag{15}$$

where dh_t is an abbreviation. This equation has the solution (using convention (f) above):

$$C_{t} = e^{\rho t} \left(C_{k-1}\ e^{-\rho (k-1)} + \int_{x=k-1}^{t} e^{-\rho x}\ dh_{x} \right) \tag{16}$$

Since taxes are paid at the end of each year, the following difference equation determines liquid assets after tax at the end of year k:

$$C_{k} = C_{k-1}\ e^{\rho} + \int_{x=k-1}^{k} e^{\rho (k-x)}\ dh_{x} - s\ \hat{M}_{k} \tag{17}$$

Solving this equation and substituting into (16) gives us the general solution:

$$C_{t} = e^{\rho t} \left(C_{0} + \int_{x=0}^{t} e^{-\rho x}\ dh_{x} - s \sum_{j=1}^{k-1} \hat{M}_{k}\ e^{-\rho j} \right) \qquad (k-1 \leqq t < k) \tag{18}$$

4. LAGRANGEAN FUNCTION AND KUHN-TUCKER CONDITIONS

For each of our equalities (13), (9), (8), (11), (10), (12), (14) ((7) follows from (11)) we introduce a non-negative Lagrangean multiplier $d\alpha_t$, β_k, ε_k, λ_k, μ_k, ξ_k and η_τ^l and form the Lagrangean:

$$L = C_T + \int_0^T C_t d\alpha_t + \sum_j [\beta_j \hat{M}_j + \varepsilon_j (1000 - v_j) + \lambda_j v_j (\hat{m}_j - \bar{m}_j - v_j) +$$

$$+ \mu_j (\bar{M}_j - \sum_l x_{jl}) - \xi_j \sum_i x_{ij} \bar{M}_j] + \sum_l \int_{\tau=-\infty}^{0} \eta_\tau^l (dG_\tau^l - \int_{t=0}^{T} dF_{\tau t}^l) \quad (19)$$

In this Lagrangean function $d\alpha_t$ and η_τ^l are generalized multipliers, the former being a non-negative multiplier density, the latter a non-negative multiplier function.

No time derivative of any variable enter explicitly into the Lagrangean. Therefore the first variation of L (in the Eulerian sense) due to a change in any function or variable reduces to the partial derivative of L with respect to that function or variable.

Introducing the abbreviation:

$$H_t = e^{\rho(T-t)} (1 + \int_{x=t}^{T} e^{\rho(x-T)} d\alpha_x) \quad (20)$$

and differentiating (19), we obtain the following Kuhn-Tucker conditions for a constrained maximum of C_T:

$$\frac{\partial L}{\partial F_{\tau t}^l} = p_t^l H_t - (p_t^l - p_\tau^l)^+ a_{\tau t} (sH_k - \beta_k - \lambda_k v_k (1 - b_{\tau t})) +$$

$$+ (p_\tau^l - p_t^l)^+ a_{\tau t} (\mu_k - \lambda_k v_k (1 - b_{\tau t}) - \xi_k \sum_j x_{jk}) -$$

$$\begin{cases} - \eta_\tau^l \leqq 0, & \text{for } \tau \leqq 0 \\ - p_\tau^l H_\tau \leqq 0, & \text{for } \tau > 0 \end{cases} \quad (21)$$

$$\frac{\partial L}{\partial v_k} = sH_k - \beta_k - \varepsilon_k - 2\lambda_k v_k + \lambda_k (\hat{m}_k - \bar{m}_k) \leqq 0 \quad (22)$$

$$\frac{\partial L}{\partial z_k} = sH_k - \beta_k - \mu_k + \xi_k \sum_j x_{jk} \leqq 0 \tag{23}$$

$$\frac{\partial L}{\partial x_{jk}} = sH_k - \beta_k - \mu_j - \xi_k(\bar{m}_k + \bar{n}_k - z_k) \leqq 0 \tag{24}$$

in which a strict inequality in either of (21) - (24) implies a zero value of the corresponding variable. Condition (24) only applies for $j<k\leqq j+6$ according to convention (b). Also we have:

$$d\alpha_t \; C_t = 0 \tag{25}$$

$$\beta_k \; \hat{M}_k = 0 \tag{26}$$

$$\varepsilon_k(1000 - v_k) = 0 \tag{27}$$

$$\lambda_k v_k(\hat{m}_k - \bar{m}_k - v_k) = 0 \tag{28}$$

$$\mu_k(\bar{M}_k - \sum_l x_{kl}) = 0 \tag{29}$$

$$\xi_k \sum_j x_{jk} \; \bar{M}_k = 0 \tag{30}$$

$$\eta^l_\tau(dG^l_\tau - \int_{t=0}^{T} dF^l_{\tau t}) = 0 \tag{31}$$

Conditions (8) - (14), (21) - (31) and non-negativity requirements on all decision variables and multipliers form the set of necessary maximizing conditions.

5. SOME PRELIMINARY CONSEQUENCES

Before examining conditions for transactions, we derive some preliminary results. An important property of H_t in (20) is that it is a strictly decreasing function of t and has a relative decrease of at least ρ. Taking the logarithm and differentiating (20) gives us:

$$\frac{\dot{H}_t}{H_t} = -\rho + \frac{\dot{I}}{I} \leqq -\rho \tag{32}$$

where I is the function:

$$I = \int_{x=t}^{T} e^{\rho(x-T)} d\alpha_x \tag{33}$$

which is non-increasing, since $d\alpha_x$ is non-negative. This function may include step changes and $\dot{I}$ must therefore be interpreted as a generalized derivative in such cases. Hence we must always have the inequality:

$$H_\tau \geqq H_t \, e^{\rho(t-\tau)} \qquad (t \geqq \tau) \tag{34}$$

The function H_t is interpreted as the overall opportunity growth factor of capital from t to T. Capital will always increase at a rate of at least ρ, but when stock investment opportunities are better, i.e. $d\alpha_t > 0$, any liquid assets C_t will be reduced to zero and $-\dot{I}/I$ will be positive showing the differential growth rate above ρ. At $t=T$ we have $H_t=1$.

As a second relation we find:

$$\hat{M}_k \bar{M}_k = 0 \tag{35}$$

From (23) we obtain that either $\beta_k>0$ or $\mu_k>0$ (or both). If $\beta_k>0$, $\hat{M}_k=0$ from (26). If $\mu_k>0$, $\bar{M}_k= \sum_l x_{kl}$. If this sum is positive we must have some $x_{kl}>0$ and by (24) $\mu_k = sH_l - \beta_l - \xi_l\bar{M}_l < sH_k$ since year l

is after k. Hence by (23), $\beta_k>0$ and $\hat{M}_k=0$. If this sum were zero, $\bar{M}_k=0$. Therefore one of taxable profits and balanced losses must be zero. This relation prevents bringing forward losses that could have been deducted when they occurred.

Also if losses shown are greater than those brought forward, i.e. $\bar{M}_k > \sum_l x_{kl}$ and therefore $\mu_k=0$, it is easily shown that taxable profits in the six years following are zero. Assume that this were not the case, i.e. that, say, $\hat{M}_l>0$. By (26) $\beta_l=0$ and by (35) $\bar{M}_l=0$. From (24) we therefore obtain $\mu_j \geqq sH_l$ for all $l-6\leqq j<l$ and so $\bar{M}_j= \sum_l x_{jl}$ contrary to hypothesis. In brief we can therefore state:

$$(\bar{M}_j - \sum_l x_{jl})\, \hat{M}_l = 0 \qquad (l-6 \leqq j < l) \tag{36}$$

In (21) the coefficients following $(p_t^l - p_\tau^l)^+ a_{\tau t}$ and $(p_\tau^l - p_t^l)^+ a_{\tau t}$ play an important role. Let these be denoted A_k and B_k respectively. As a first result we prove that the second coefficient obeys:

$$B_k = \mu_k - \xi_k \sum_j x_{jk} - \lambda_k v_k(1-b_{\tau t}) \geqq 0 \tag{37}$$

As a hypothesis, assume the contrary. Then either $\xi_k \sum_j x_{jk} > 0$ $\lambda_k v_k(1-b_{\tau t})$, or both. Assume at first that $\lambda_k v_k>0$. Then by (22) and (28) $sH_k - \beta_k \geqq \lambda_k v_k$ which by (23) contradicts hypothesis. Assume instead $\xi_k \sum_j x_{jk} > 0$ and $\lambda_k v_k=0$. Then by (30) $\bar{M}_k=0$ and by (24) $\mu_j = sH_k - \beta_k$ for some j. Since $\mu_j\geqq 0$ this consequence by (23) also contradicts hypothesis. Hence (37) must be valid.

A similar result applies to the former coefficient A_k in (21) namely that in an optimal solution we may always choose:

$$A_k = sH_k - \beta_k - \lambda_k v_k(1-b_{\tau t}) \geqq 0 \tag{38}$$

Assume by hypothesis the contrary. If $\lambda_k v_k(1-b_{\tau t}) > 0$ then by (28) we have a strict inequality in (22) and $v_k=0$ contrary to assumption. Hence $\lambda_k v_k=0$ and by hypothesis $sH_k - \beta_k < 0$. This would imply by (24) that $x_{jk}=0$ for all relevant j. Therefore we would have strict inequalities in (22), (23) and (24) and so $v_k=z_k=x_{jk}=0$. Since then

$\hat{m}_k + \hat{n}_k = 0$, there can be no $dF^l_{\tau t}>0$ for any $p^l_t > p^l_\tau$. We are therefore free to adjust A_k upwards without affecting (21) or any other of our conditions and this is permissible at least until this expression becomes zero. Hence among the set of optimal solutions we can always choose our variables to comply with (38). Introducing (38) as an additional constraint would therefore have no effect on the solution. Adopting this, we obtain specifically:

$$\beta_k \leqq sH_k \tag{39}$$

By a similar reasoning we find that we may always choose the coefficient following $(p^l_\tau - p^l_t)^+ a_{\tau t}$ in (21) to obey:

$$B_k = \mu_k - \xi_k \sum_j x_{jk} - \lambda_k v_k (1-b_{\tau t}) \leqq sH_k \tag{40}$$

By hypothesis assume the contrary. Then $\mu_k > sH_k > sH_l - \beta_l - \xi_l \bar{M}_l$ and μ_k may be lowered without violating (23) or (24). Due to the strict inequality in (23) we have $z_k=0$ and $\mu_k>0$ then implies $\bar{m}_k + \bar{n}_k - \sum_l x_{kl} = 0$ by (29). If $x_{kl}>0$ for some relevant l, (24) yields an equality contrary to hypothesis. Hence we would have $\bar{m}_k + \bar{n}_k = 0$ which implies that $dF^l_{\tau t}=0$ for prices $p^l_\tau > p^l_t$. Thus the left-hand member of (40) may be lowered without affecting any of our constraints. By including the inequality (40) as an additional constraint, this could never change the optimal solution.

6. PURCHASING AND SELLING POLICIES

As a first result we derive:

$$\eta_\tau^l > 0 \qquad (\tau \leqq 0) \tag{41}$$

which implies that all initial holdings at some time should be sold. In (21) we have $p_t^l > (p_t^l - p_\tau^l)^+$ and $H_t > sH_k\, a_{\tau t}$, since $s\, a_{\tau t} < 1$ and k follows t in time. All other terms are non-negative by (37). The inequality (41) follows directly. All initial holdings must therefore always be sold even if they might have to be sold at a loss. This is a self-evident result.

We now take a second look at (21) for $\tau > 0$. Assume that price has not increased from τ to t for some share l, i.e. $(p_\tau^l - p_t^l) \geqq 0$. Then we have:

$$p_\tau^l\, (H_\tau - B_k\, a_{\tau t}) \geqq p_t^l\, (H_t - B_k\, a_{\tau t}) \tag{42}$$

Since $a_{\tau t} \leqq 1$, and B_k may be chosen to comply with (40), $H_\tau > H_t$ and $H_t \geqq H_k$, we must have a strict inequality in (42). This implies the important result that <u>shares bought at $\tau > 0$ may never be sold at a loss</u>. It therefore never pays to arrange for a loss deliberately. This result was already proven in a slightly different manner in [2].

The function H_τ is restricted from below by (21) and (34). We can therefore choose it according to the recursive relation:

$$H_\tau = \underset{l,\, t>\tau}{Max} \left\{ \frac{p_t^l}{p_\tau^l} H_t - \left(\frac{p_t^l}{p_\tau^l} - 1\right)^+ a_{\tau t}\, A_k + \left(1 - \frac{p_t^l}{p_\tau^l}\right)^+ a_{\tau t}\, B_k,\quad H_t e^{\rho(t-\tau)} \right\} \tag{43}$$

where the convention (f) is adopted. Due to the strict inequality in (42) for any $p_\tau^l \geqq p_t^l$ such an l and t will never maximize the argument in (43). Hence we always have:

$$H_\tau = \underset{l,\, t>\tau}{Max} \left\{ \frac{p_t^l}{p_\tau^l} H_t - \left(\frac{p_t^l}{p_\tau^l} - 1 \right)^+ a_{\tau t} A_k, \quad H_t\, e^{\rho(t-\tau)} \right\} \tag{44}$$

and $H_T = 1$. From (21), for $\tau \leqq 0$ we must then choose:

$$\eta_\tau^l = \underset{t>0}{Max} \left\{ p_t^l H_t - (p_t^l - p_\tau^l)^+ a_{\tau t} A_k + (p_\tau^l - p_t^l)^+ a_{\tau t} B_k \right\} \tag{45}$$

Equation (44) governs the times at which stock may be purchased and sold, since $dF_{\tau t}^l > 0$ requires equality in (21) and for each τ corresponds to the maximizing indices l, t in (44). The equation also governs when capital has to be spent, i.e. when shares have to be purchased. At times τ when H_τ undergoes a relative decrease in excess of ρ, according to (32) - (33) this will require a $d\alpha_\tau > 0$ and thus $C_\tau = 0$ from (25). If there is any liquid capital available prior to such a point, it therefore has to be spent in its entirety.

Similarly, (45) defines the points in time at which initial stockholdings have to be sold. In this case (41) requires that all stock should be sold at some time and a $dF_{\tau t}^l > 0$ demands equality in (21) corresponding to a maximizing index t in (45).

If all A_k and B_k were known for all years following τ, it would be a relatively simple task to fold out (44) and (45), which directly would yield the optimal policy. This would be the case if taxes were negligable, since $s = 0$ implies $A_k = B_k = 0$ for all k. For $s > 0$, however, the coefficients A_k and B_k will depend on the profit/loss situation for each year in question.

Some cases may illustrate this dependence. When there are no profits on older shares $\hat{m}_k = 0$ or at least no profits above current similar losses $\hat{m}_k - \bar{m}_k \leqq 0$, we must have $\varepsilon_k = \lambda_k v_k = 0$ according to (27) - (28) and $A_k = 0$, $\beta_k = sH_k$ according to (22), (38). Similarly, when there is a net loss shown that cannot be used for later deduction purposes, i.e. $\bar{M}_k > \sum_l x_{kl}$, $A_k = 0$ according to (23), (38).

Another similar case applies when some but not all balanced losses in one of the preceding six years have been deducted in year k.

The cases above require A_k to take on its minimum value zero. The opposite situation arises when there are large profits. If $\hat{M}_k > 0$, then $\beta_k = 0$. For younger shares $b_{\tau t} = 1$ and $A_k = sH_k$ from (38). For older shares, if their net profits $\hat{m}_k - \bar{m}_k$ are in excess of 1000, (27) - (28) require $\lambda_k v_k = 0$ and $A_k = sH_k$ by (38). Since a greater A_k lowers the corresponding argument in (44), the requirements for sales making profits up to 1000 net are less restrictive for older shares than for younger shares, which also is in accordance with intuition. Conversely, if $A_k = sH_k$ then $\beta_k = \lambda_k v_k (1 - b_{\tau t}) = 0$ and $\varepsilon_k > 0$ by (22). This requires $v_k = 1000$ and $\hat{m}_k - \bar{m}_k \geqq 1000$ by (27) and (11).

When taxes have to be taken into consideration there is therefore a more or less strong dependence between the sequences of A_k and B_k determining the H_τ and η_τ^l by (44) - (45) and the profits and losses resulting from (44) - (45) influencing the same sequences. A heuristic, iterative procedure, similar to the one applied in [2] would be to start out by choosing all A_k maximal and all B_k minimal. After computing H_τ and η_τ^l and the resulting transactions, the profit/loss situation for each year is determined. When there is a conflict between values of A_k, B_k and profits/losses in year k, A_k is lowered and/or B_k raised by the minimum amount necessary to avoid this conflict according to (21) and the procedure computing H_τ is repeated. It seems plausible, although no proof is offered here, that such a procedure might converge.

In [2] three categories of price increases were defined:

Weak price increase.

$$\frac{p_t}{p_\tau} \leqq e^{\rho(t-\tau)} \qquad (46)$$

Medium price increase.

$$e^{\rho(t-\tau)} < \frac{p_t}{p_\tau} \leqq \frac{e^{\rho(t-\tau)} - sa_{\tau t}}{1 - sa_{\tau t}} \qquad (47)$$

Strong price increase.

$$\frac{p_t}{p_\tau} > \frac{e^{\rho(t-\tau)} - sa_{\tau t}}{1 - sa_{\tau t}} \tag{48}$$

Due to the continuous time variable in our present model, there are no strict conclusions following from distinctions between medium and strong increases as was given in [2] except when t coincides with k, i.e. at the last date of the year. However, for weak increases capital should be invested at the opportunity rate ρ rather than in shares, the same obvious conclusion as in [2].

7. EXAMPLE

Consider a case in which there are two kinds of shares, the prices of which follow the time development:

$$\begin{cases} p_t^1 = 100\, e^{0.1t} \\ p_t^2 = 50\, e^{0.08t} \end{cases} \tag{49}$$

The time $t = 0$ is December 31/January 1, Year 1. The second kind of shares has experienced a sharp price decline recently. A person has an initial holdings of 500 shares of the second kind that were bought on July 1, Year -1 for 37,500 Sw.Cr. and an initial cash balance amounting to 100,000 Sw.Cr. The opportunity rate of interest is 0.03 per year and falls below the after-tax growth rate $(1-s)0.1$ of the shares of the first kind for the tax rate $s = 0.6$. The wealth of the person at the end of Year 3 is to be maximized.

Our heuristic procedure suggests that the three A_k-coefficients should be given their maximal values sH_k initially. However, $A_1 > 0$ and $A_2 > 0$ would require that v_1, $v_2 > 0$, which is impossible since there are no older shares available to be sold at a profit in years 1 and 2. Let our initial coefficients therefore be chosen as $A_1 = A_2 = 0$ and $A_3 = sH_3 = 0.6$. Since shares of the first kind have the highest relative growth, their price will govern the development of the function H_t.

Given the values of the A_k as above, the solution to (44) will be:

$$H_t = \begin{cases} e^{0.1(3-t)} & t \leq 2 \\ 0.4\, e^{0.1(3-t)} + 0.6 & 2 < t \leq 3 \end{cases} \tag{50}$$

At all times we have $\dot{H}_t/H_t < -\rho = -0.03$ requiring that cash should never be held. At $\tau = 0$, the maximizing value of t in the right-hand member of (44) is $t = 3$, indicating that all stock purchased at $\tau = 0$ should be sold at $t = 3$, i.e. kept throughout.

The time at which the initial holdings are to be sold is governed by (45):

$$n^2_{-3/2} = \underset{t>0}{Max} \{p^2_t H_t + (75 - p^2_t) a_{-3/2\ t} B_k\} \tag{51}$$

With p^2_t given by (49) and H_t by (50) the right-hand member in (51) will be maximized by $t = 0+$, independently of the values of $a_{-3/2\ t}$ and B_k, giving us:

$$n^2_{-3/2} = 50\ e^{0.3} + 25\ B_1 \tag{52}$$

The initial holdings should therefore be sold as early as possible in Year 1 (and not be kept until June 30, the latest date before $a_{-3/2\ t}$ falls to 0.4) . This transaction adds 25,000 Sw.Cr. to the initial capital which should be spent immediately on stock of the first kind. It also creates a loss in Year 1 amounting to $\bar{n}_1$ = 12,500 Sw. Cr.

There will be no current losses in Year 3 and $\beta_3 = 0$, since $A_3 = sH_3 = 0.6$. From (24) we thus obtain $\mu_1 > 0$ and $\mu_2 > 0$. Eq.(29) then requires $\bar{n}_1$ to be brought forward for deductions in the six-year period following, i.e. $x_{12} + x_{13} = \bar{n}_1$. In Year 2 we have $\beta_2 = sH_2$ since $A_2 = 0$ and $v_2 = 0$. Therefore $x_{12} > 0$ is impossible according to (24). Hence $x_{13} = \bar{n}_1$ and $\mu_1 = sH_3 = 0.6$.

For our initial cash and the revenues from selling off the initial holdings we can buy 1,250 shares of the first kind for 125,000 Sw.Cr. These will be sold at $t = 3$ creating a gross reduced profit on older shares amounting to $\hat{m}_3 = 0.4 \cdot 1250(100\ e^{0.3} - 100) =$ = 17,493 Sw.Cr.

Since $\beta_3 = 0$, from (22) we obtain that either ε_3 or $\lambda_3 v_3$ is positive, the former condition requiring $v_3 = 1000$ by (27) and the latter $v_3 = 17,493$ by (28) violating (8). Hence $v_3 = 1000$ and taxable profits will be $\hat{M}_3 = \hat{m}_3 - v_3 - x_{13}$ = 17,493 - 1,000 - - 12,500 = 3,993 Sw. Cr. Taxes will be 2,396 Sw. Cr.

The maximal wealth at the end of Year 3 thus amounts to $C_3 = 1250 \cdot 100 \cdot e^{0.3} - 2396$ = 166,336 Sw. Cr. This corresponds to an overall growth of 9.5 per cent per annum after taxes.

8. CONCLUSIONS

The model described in the foregoing offers an analytical instrument for studying the optimal purchasing and selling policy for stock when taking the basic Swedish tax legislation into account. It is evident that the prevailing tax rules to a considerable degree are responsible for the complexity in the model and in a rational decision making in this field. If the standard deduction had been applicable to profits independently of the age of the stock sold, or if the profit reduction factor had been a continuously decreasing function of $t - \tau$, e.g. an exponential decay, or if older loss deductions were not limited to a six-year period, the model would have gained a considerable amount of simplicity.

What appears to be of highest priority for improving the model in its present form is to develop a strict algorithm for choosing the A_k- and B_k-coefficients. Once such a procedure is devised, a computerized version would be easy to implement. Other improvements would be gained by developing a procedure for sensitivity analysis as to changes in forecasted prices or growth rates and also conditions for myopic decision rules to be consistent with the optimal policy. Also of interest would be to include the possibility to acquire additional funds (e.g. saving from other income) and to take capital gains on other assets than shares into consideration.

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